The Special Lecture Series

Okinawa Institute of Sience and Technology, Japan

Slicing and fine properties for functions with Bounded $\mathcal{A}$-variation

Adolfo Arroyo-Rabasa

The University of Warwick, United Kingdom

Program

Let $\Omega \subset \mathbb{R}^n$ be an open domain. We study the slicing and fine properties of functions of anisotropic BV-spaces. Namely, for the space

$$\mathrm{BV}^{\mathcal{A}}(\Omega) := \left\{ u \in \mathrm{L}^1(\Omega; V) : \mathcal{A}u \in \mathcal{M}(\Omega; W) \right\},$$

of functions with bounded $\mathcal{A}$-variation. Here, V and W are finite dimensional euclidean spaces which, up to a linear isomorphism may be thought of as $\mathbb{R}^N$ and $\mathbb{R}^M$. In order to keep the exposition as simple as possible, I will restrict to the slicing and fine properties of the spaces where $\mathcal{A}$ is a constant coefficient first-order homogeneous linear differential operator of the form

$$\mathcal{A} = \sum_{j=1}^{n} A_j \partial_j , \qquad A_j \in \mathrm{Lin}(V; W).$$

The purpose of this work is to give a comprehensive determination of the structural and fine properties of functions in $\mathrm{BV}^{\mathcal{A}}(\Omega)$, very much in the fashion of what is known for $\mathrm{BV}(\Omega; \mathbb{R}^N)$. Our main result is a characterization of all operators $\mathcal{A}$ (in the form of a closed algebraic property) satisfying the following one-dimensional structure theorem: every $u \in \mathrm{BV}^{\mathcal{A}}$ can be sliced into one-dimensional BV-sections. Moreover, decomposing $\mathcal{A}u$ into an absolutely continuous part $\mathcal{A}^a u$, a Cantor part $\mathcal{A}^c u$ and a jump part $\mathcal{A}^j u$, each of these measures can be recovered from the corresponding classical D^a, D^c and D^j BV-derivatives of its one-dimensional sections. By means of this result, we are able to analyze the set of Lebesgue points as well as the set of jump points where these functions have approximate one-sided limits. Thus, proving a structure and fine properties theorem in $\mathrm{BV}^{\mathcal{A}}$. Our results extend most of the classical fine properties of BV (and all of those known for BD). In particular, we establish a slicing theory and fine properties for $\mathscr{BV}^k$, BD^k and a whole class of $\mathrm{BV}^{\mathcal{A}}$-spaces that is not covered by the existing theory.

Lecture I (7.04.21 9:00 CET)

Introduction and characterization of slicing properties

In this first lecture, I will give a brief outlook of the classical slicing and fine properties of Sobolev functions, BV functions and functions of bounded deformation. These include some basic measure theoretic background and definitions such as one-dimensional slices, approximate continuity and the jump set of an integrable function.

After we have revised the classical background theory, I will introduce the set-up of spaces with $\mathcal{A}$-variation ($\mathrm{BV}^{\mathcal{A}}$-spaces). The rest of the lecture will be devoted to state and discuss the first main result: the equivalence between the $\mathrm{rank}_{\mathcal{A}}$-one property

$$\bigcap_{\substack{\pi \le \mathbb{R}^n \\ \dim(\pi)=n-1}} \operatorname{span}\{\operatorname{Im}\mathbb{A}(\xi) : \xi \in \pi\} = \{0_W\}, \qquad \mathbb{A}(\xi) := \sum_{j=1}^n A_j \xi_j,$$

and the $\mathrm{BV}^{\mathcal{A}}$-spaces that admits a one-dimensional sectional representation.

Lecture II (14.04.21 9:00 CET)

Proof of the slicing characterization and statement of the structure theorem

The first half of this lecture will be devoted to the proof the equivalence between the $\mathrm{rank}_{\mathcal{A}}$-one property and the one-dimensional slicing properties of $\mathrm{BV}^{\mathcal{A}}$-spaces. Once we have reached a proper understanding of the slicing result, I will proceed to discuss some applications oriented to the understanding of the geometric properties of A-gradients (dimensional and rectifiability properties).

The second half of this lecture will focus on the statement and background required to discuss the second main result: the structure and fine properties theorem for $\mathrm{BV}^{\mathcal{A}}$ spaces satisfying the rank-one property. After stating and formally discussing this, I will spend some time to discuss several examples. During the reminder of the lecture I will introduce algebraic and *slicing theory constructions* that will serve as foundation stones for the proof of the Structure theorem.

Lecture III (21.04.21 9:00 CET)

Proof of the structural and fine properties, higher-order operators

The third lecture will be focused on discussing the proofs the second main result, these include

1. the analysis of Lebesgue point properties,
2. the proof of a sectional structure theorem,
3. and the proof of the structural and fine properties that follow from (1) and (2).

Finally, if time permits, I will discuss some of the considerations that are required to apply the same techniques for operators of arbitrary order.