Lecture 1 : Vector Analysis

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I. INTRODUCTION

A. Definition and Notations

In 3-dimension Euclidean space, a quantity which requires both direction and magnitude to specify is called a vector. On the other hand, a quantity with which one can describe completely using magnitude is called a scalar.

Example : While the position of an object in 3-dim Euclidean space is a vector, its weight is a scalar.

- Notation : we will ues letters with a arrow on top of them to denote vectors and letters without any decoration to represent scalars.
  - A vector : $\vec{A}$.
  - A scalar : a.

B. Operations on Vectors

As everyone is familiar with, once a origin and a basis (say $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$) is chosen in 3-dim Euclidean space, a vector (in this 3-dim Euclidean space) is completely determined by 3 scalars. Specfically, any vector in 3-dim Euclidean space can be written as

$$\vec{A} = (A_1, A_2, A_3), \tag{1}$$

here A_i with $i \in \{1, 2, 3\}$ are scalars. The magnitude of a vector $\vec{A}$ which is denoted by $|\vec{A}|$ is defined by $|\vec{A}| = \sqrt{\sum_{i=1}^{3} A_i^2}$.

Example : Let $\vec{A} = (2, 1, -1)$. then $|\vec{A}| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$.

A vector $\vec{A}$ is called an unit vector if $|\vec{A}| = 1$. For example, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ are all unit vectors.

Addition and Substraction of Vectors

Given 2 vectors $\vec{A} = (A_1, A_2, A_3)$ and $\vec{B} = (B_1, B_2, B_3)$, one can define addtion and substraction of $\vec{A}$ and $\vec{B}$ through

$$\begin{aligned} \text{Addition} : \vec{C} &= \vec{A} + \vec{B} = (A_1 + B_1, A_2 + B_2, A_3 + B_3), \\ \text{Substraction} : \vec{C'} &= \vec{A} - \vec{B} = (A_1 - B_1, A_2 - B_2, A_3 - B_3). \end{aligned} \tag{2}$$

These rules we mentioned about "+" and "−" apply to more than 2 vectors as well : Let $\vec{A} = (A_1, A_2, A_3)$, $\vec{B} = (B_1, B_2, B_3)$ and $\vec{C} = (C_1, C_2, C_3)$. Then one has

$$\begin{aligned} &\vec{A} + \vec{B} + \vec{C} = (A_1 + B_1 + C_1, A_2 + B_2 + C_2, A_3 + B_3 + C_3), \\ &\vec{A} - \vec{B} + \vec{C} = \vec{A} + \vec{C} - \vec{B} = (A_1 - B_1 + C_1, A_2 - B_2 + C_2, A_3 - B_3 + C_3). \end{aligned} \tag{3}$$

Notice the addition operator "+" on vectors is associated, namely one has

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}. \tag{4}$$

How about the substraction operator "−" on vectors? Is it true that

$$\vec{A} - (\vec{B} - \vec{C}) = (\vec{A} - \vec{B}) - \vec{C}? \tag{5}$$

Multiplying a vector by a scalar

Given a scalar a and a vector $\vec{A} = (A_1, A_2, A_3)$, one can define the multiplication of a vector by a scalar (which is again a vector) by

$$\vec{C} = a\vec{A} = (aA_1, aA_2, aA_3). \tag{6}$$

Notice by eqs. (6), the following distributive law holds

$$a(\vec{A} \pm \vec{B} \pm .... \pm \vec{C}) = a\vec{A} \pm a\vec{B} \pm .... \pm a\vec{C}. \tag{7}$$

Example : Let $\vec{A} = (2, 1, 0)$, $\vec{B} = (1, -1, 2)$ and $a = 2.0$, then we have

$$\begin{aligned} \vec{A} + \vec{B} &= (3, 0, 2) \\ \vec{A} - \vec{B} &= (1, 2, -2) \\ a\vec{B} &= (2, -2, 4) \\ a(\vec{A} + \vec{B}) &= (6, 0, 4) \\ a\vec{A} \ + \ a\vec{B} &= (4, 2, 0) + (2, -2, 4) = (6, 0, 4) \end{aligned} \tag{8}$$

Notice every vector $\vec{A} = (a_1, a_2, a_3)$ can be written as $\vec{A} = a_1 e_1 + a_2 e_2 + a_3 e_3$. Hence the set of unit vectors e_i is called a basis of the 3-dim Euclidean (vector) space.

Scalar product of 2 vectors : geometrical definition

Given 2 vectors $\vec{A}$ and $\vec{B}$, one can define the scalar product between these 2 vectors which will produce a scalar as follows

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta, \tag{9}$$

here θ stands for the angle between vectors $\vec{A}$ and $\vec{B}$. Notice $|\vec{A}| \cos\theta$ is the projection of $\vec{A}$ onto $\vec{B}$.

Example : If $\vec{A} \cdot \vec{B} = 0$ and $|\vec{A}| \neq 0$, $|\vec{B}| \neq 0$, then $\vec{A}$ is perpendicular to $\vec{B}$.

Next, for e_i, one has

$$e_i \cdot e_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \tag{10}$$

Further one can show that the scalar product between 2 vectors is commutative : $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$.

Now using the observation that $|\vec{A}| \cos\theta$ in $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$ can be interpretated as the projection of $\vec{A}$ onto $\vec{B}$, one has

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}. \tag{11}$$

By induction the distributive law holds as well for the operation of scalar product on vectors :

$$\vec{A} \cdot (\vec{B} \pm \vec{C} \pm ....) = \vec{A} \cdot \vec{B} \pm \vec{A} \cdot \vec{C} \pm .... \tag{12}$$

Finally with all above results, using $\vec{A} = a_1e_1 + a_2e_2 + a_3e_3$ and $\vec{B} = b_1e_1 + b_2e_2 + b_3e_3$, one would reach another expression for $\vec{A} \cdot \vec{B}$

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i B_i. \tag{13}$$

Cross product of 2 vectors

Given 2 vectors $\vec{A}$ and $\vec{B}$, the vector product of $\vec{A}$ and $\vec{B}$ is again a vector $\vec{C}$ which is perpendicular to both $\vec{A}$ and $\vec{B}$. Further, $\vec{A}$, $\vec{B}$ and $\vec{C}$ form a right-handed system. Specifically one has

$$\vec{C} = \vec{A} \times \vec{B} \quad \text{with} \quad |\vec{C}| = |\vec{A}||\vec{B}| \sin\theta, \tag{14}$$

here θ is the angle between $\vec{A}$ and $\vec{B}$. Geometircally, the vector product of 2 vectors $\vec{A}$ and $\vec{B}$ produces another vector $\vec{C}$ (with magnitude given by $|\vec{C}| = |\vec{A}||\vec{B}| \sin\theta$) which is perpendicular to the plan spanned by $\vec{A}$ and $\vec{B}$. Further the 3 vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ forms a right-handed system.

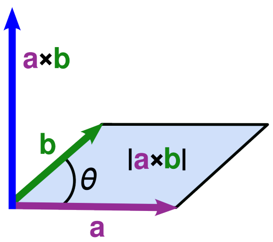

FIG. 1: Cross product of vectors $\vec{a}$ and $\vec{b}$. Source : Wiki, User:Acdx

Example : If $\vec{A} \times \vec{B} = 0$ and $|\vec{A}| \neq 0$, $|\vec{B}| \neq 0$, then $\vec{A}$ is parallel to $\vec{B}$, namely there exists a nonzero scalar c so that $\vec{A} = a\vec{B}$.

For e_i, one can easily show

$$e_i \times e_j = \epsilon_{ijk} e_k, \tag{15}$$

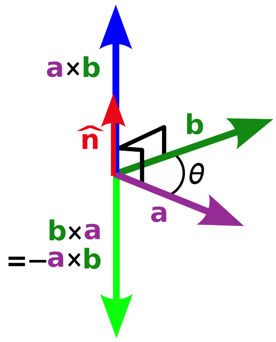

FIG. 2: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. Source : Wiki, User:Acdx

here ϵ_{ijk} is called Levi-Civita symbol and is given by

$$\epsilon_{ijk} = \begin{cases} 1 & ijk \text{ is an even permutation of } 123 \\ -1 & ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise} \end{cases} . \tag{16}$$

As a result, one sees that $\vec{C} = \vec{A} \times \vec{B}$ can be written as

$$\vec{C} = \sum_i C_i e_i = \sum_i (\sum_{j,k} \epsilon_{ijk} A_j B_k) e_i, \tag{17}$$

or specifically, one has

$$C_1 = A_2 B_3 - A_3 B_2,\ C_2 = A_3 B_1 - A_1 B_3,\ C_3 = A_1 B_2 - A_2 B_1. \tag{18}$$

It is easy to see from eq. 17 and eq. 18 that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

Conventionally, the vector $\vec{C}$ $(= \vec{A} \times \vec{B})$ can be represented by a determinant

$$\vec{C} = \begin{vmatrix} e_1 & e_2 & e_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} . \tag{19}$$

One can define triple scalar and triple vector products of 3 vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ similarly.

For example, the scalar triple product $\vec{A}\cdot(\vec{B}\times\vec{C})$ is given by

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \tag{20}$$

Geometrically, the scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c})$ is the (signed) volume of the parallelepiped defined by the three vectors given.

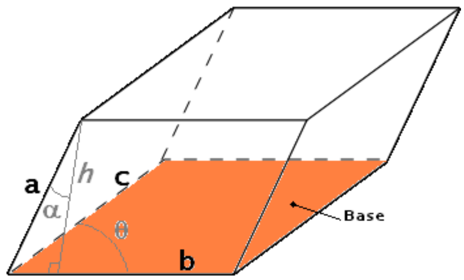

FIG. 3: Geometric interpretation of the scalar triple product. Source : Wiki, by Baard Johan Svensson

Example : Let $\vec{A} = (1, 2, -1)$, $\vec{B} = (0, 1, 1)$ and $\vec{C} = (1, -1, 0)$. Let $\vec{D} = \vec{B}\times\vec{C}$ and $\vec{F} = \vec{A}\times(\vec{B}\times\vec{C})$, then one has

$$\vec{D} = \sum_{i=1}^{3}\sum_{j,k}\epsilon_{ijk}B_jC_ke_i = (B_2C_3 - B_3C_2)e_1 + (B_3C_1 - B_1C_3)e_2 + (B_1C_2 - B_2C_1)e_3$$
$$= e_1 + e_2 - e_3 = (1, 1, -1) \tag{21}$$

and

$$\vec{F} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & -1 \\ 1 & 1 & -1 \end{vmatrix} = -e_1 - e_3 = (-1, 0, -1). \tag{22}$$

What we have learned so far regarding vectors in 3-dimension Euclidean space can be generalized to vectors in n-dimension Euclidean space. Specifically each vector$\vec{A}$ in n-dimension Euclidean space is determined by n scalars : $\vec{A} = (a_1, a_2, a_3, ..., a_n)$. Operations on vectors in n-dimension Euclidean space are the same as those for vectors in 3-dimension

Euclidean space. For example, let $\vec{A} = (a_1, a_2, a_3, ..., a_n)$ and $\vec{B} = (b_1, b_2, b_3, ..., b_n)$ be 2 vectors in n-dimension Euclidean space, then the inner product of $\vec{A}$ and $\vec{B}$ is given by $\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 + ... + a_n b_n$. Using Levi-Civita symbol, can you come up with an expression for cross product of 2 vectors in n-dimension Euclidean space?

II. GRADIENT ∇

Before we proceed, at the moment what we have learned is vectors and scalars in 3-dim Euclidean space. Actually one can also assign each point in the 3-dim Euclidean space a vector (or a scalar). With such assignment one constructs a vector field (scalar field) in 3-dime Euclidean space. Unless made explicitly, we will assume that vector and scalar fields considered in this lecture have continuous derivatives.

Example : $\vec{A}(x, y, z) = (x, xy, xz)$ ($\varphi(x, y, z) = x^2 yz$) is a vector field (scalar field) in 3-dim Euclidean space .

With vector and scalar fields in 3-dim Euclidean space, one can differentiate and integrate vector and scalar fields componetwise.

Example : Let $\vec{A}(x, y, z) = (x, xy, xz)$, then we have $\frac{\partial \vec{A}}{\partial x} = (1, y, z)$.

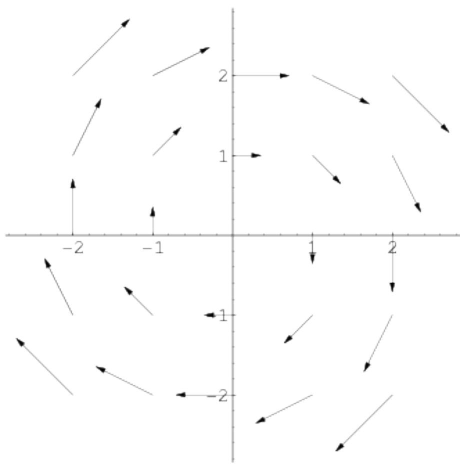

FIG. 4: A 2-dim vector field : f(x,y) = (y,-x). Source : http://www.math.umn.edu/ nykamp/m2374/readings/vecfield/

A. Definition

Definition of the operator ∇

Let $\phi(x_i)$ be a scalar function in 3-dim Euclidean space, namely $\phi'(x'_i) = \phi(x_i)$ for any rotational coordinate system $\vec{x'}$ from the original coordinate system $\vec{x}$. Then the operator ∇ takes a scalar field to a vector field. Specifically we have

$$\nabla\phi(x_1, x_2, x_3) = \sum_{i=1}^{3} \frac{\partial \phi}{\partial x_i} e_i. \tag{23}$$

From above definition one see that ∇ is a vector field and one has

$$\nabla = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} e_i. \tag{24}$$

Gradient of a scalar is important in physics in expressing the relation between a conservative force $\vec{F}$ (gravitational and electrostatic) and its potential V

$$\vec{F} = -\nabla V. \tag{25}$$

Notice we have

$$dV(x_1, x_2, x_3) = \frac{\partial V}{\partial x_1} dx_1 + \frac{\partial V}{\partial x_2} dx_2 + \frac{\partial V}{\partial x_3} dx_3 = \nabla V \cdot d\vec{r} = -\vec{F} \cdot d\vec{r}. \tag{26}$$

In other word, we see the physical meaning of difference of potentials is energy or work.

Eaxmple : Let $V(r) = V(\sqrt{x^2 + y^2 + z^2})$, then we have

$$\nabla V(r) \;=\; \frac{\partial V(r)}{\partial x} e_1 + \frac{\partial V(r)}{\partial y} e_2 + \frac{\partial V(r)}{\partial z} e_3 \tag{27}$$

Now using $\frac{\partial r}{\partial x_i} = \frac{x_i}{\sqrt{\sum x_j^2}} = \frac{x_i}{r}$, one arrives at

$$\begin{aligned} \nabla V(r) &= \sum_i \frac{\partial V(r)}{\partial x_i} e_i = \sum_i \frac{dV(r)}{dr} \frac{\partial r}{\partial x_i} e_i \\ &= \frac{dV(r)}{dr} \frac{\vec{r}}{r}. \end{aligned} \tag{28}$$

Geometrical interpretation of ∇

Let C be a constant and S_C be a surface defined by $S_C(x_i) = C$. Further let $d\vec{r}$ be a very tiny vector moving along the surface S_C. Then one finds $\nabla S_C \cdot d\vec{r} = dS_C = 0$, namely ∇S_C is perpendicular to the constant surface S_C.

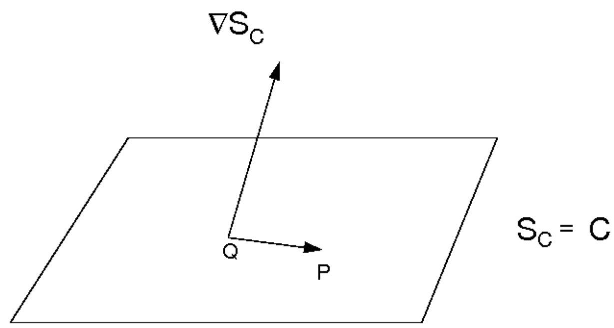

FIG. 5: Geometric interpretation of gradient.

Let C_1 and C_2 be 2 adjacent constant surface and let $d\vec{r}$ be a vector moving from C_1 to C_2. Then one has

$$C_1 - C_2 = dS_C = (\nabla S_C) \cdot d\vec{r}. \tag{29}$$

One further sees that for a given $d\vec{r}$, dS_C is maximum when $d\vec{r}$ is parallel to ∇S_C. Hence ∇S_C is a vector having the direction of the maximum space rate of change of S_C.

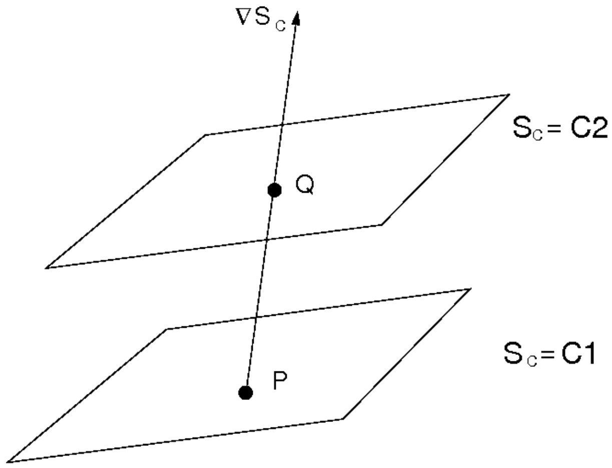

FIG. 6: Gradient.

Example : Let $\varphi(x, y, z) = (x^2 + y^2 + z^2)^{1/2} = r = C$, namely φ is a sphere with radiu r. Then one finds $\nabla\varphi(r) = \vec{r}/r$. In other word, the gradient is in the radial direction and is normal to the surface of sphere C.

III. DIVERGENCE $\nabla\cdot$

A. Definition

Definition of the operator $\nabla\cdot$

Let $\vec{V}$ be a vector field in 3-dim Euclidean space. Then the operator $\nabla\cdot$ takes a vector field to a scalar field. Specifically we have

$$\nabla \cdot \vec{V} = \sum_{i=1}^{3} \frac{\partial V_i}{\partial x_i}. \tag{30}$$

Example : let $\vec{A} = \vec{r}V(r)$, then one has

$$\nabla \cdot \vec{A} = \sum_i \frac{x_i V(r)}{\partial x_i}$$
$$= 3V(r) + \sum_i x_i \frac{V(r)}{\partial x_i}$$
$$= 3V(r) + r\frac{dV(r)}{dr} \quad (31)$$

Notice if we let $V(r) = r^{n-1}$, then

$$\nabla \cdot (\vec{r}r^{n-1}) = (n+2)r^{n-1} \quad (32)$$

This divergence is zero for $n = -2$ except at the origin ($r = 0$).

Geometrical interpretation of $\nabla\cdot$

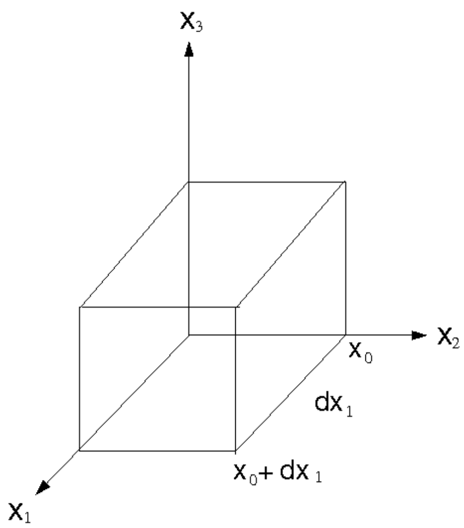

FIG. 7: Differential rectangular parallelepiped.

Let $\vec{v_i}$ be the velocity of a compressible fluid and $\rho(x_i)$ be its density at point $\vec{x}$. If we consider a small volume $dx_1 dx_2 dx_3$, then the fluid flowing into and flowing out this volume per unit time (perpendicular to the surface $dx_2 dx_3$) are given by $\rho v_{x_1}|_{x=0} dx_2 dx_3$ and $\rho v_{x_1}|_{x=dx_1} dx_2 dx_3$, respectively. Notice using

$$\rho v_{x_1}|_{x=dx_1} = \Big[\rho v_{x_1} + \frac{\partial(\rho v_{x_1})}{\partial x_1} dx_1\Big]_{x_1=0} dx_2 dx_3, \tag{33}$$

One sees that the net out flow perpendicular to the surface $dx_2 dx_3$ is given by

$$\text{Net out flow} \perp dx_2 dx_3 \text{ at } x_1 = \Big[\frac{\partial(\rho v_{x_1})}{\partial x_1}\Big]_{x_1=0} dx_1 dx_2 dx_3 \tag{34}$$

Applying above arguments to other 2 surfaces, one see that the net flow out of the volume $dx_1 dx_2 dx_3$ per unit time is given

$$\text{Net flow out (per unit time)} = \nabla \cdot (\rho \vec{v}) dx_1 dx_2 dx_3. \tag{35}$$

A vector field $\vec{B}$ satisfies $\nabla \cdot \vec{B} = 0$ is called solenoidal field. We will come back to this when discussing Maxwell's equations.

IV. CURL $\nabla\times$

A. Definition

Definition of the operator $\nabla\times$

Let $\vec{V}$ be a vector field in 3-dim Euclidean space. Then the operator $\nabla\times$ takes a vector field to another vector field. Specifically we have

$$\begin{aligned} \nabla \times \vec{V} &= (\frac{\partial V_2}{\partial x_3} - \frac{\partial V_3}{\partial x_2}) e_1 + (\frac{\partial V_3}{\partial x_1} - \frac{\partial V_1}{\partial x_3}) e_2 \\ &+ (\frac{\partial V_1}{\partial x_2} - \frac{\partial V_2}{\partial x_1}) e_3. \end{aligned} \tag{36}$$

From the definition of curl, for a scalar f and a vector $\vec{K}$, one can easily prove

$$\nabla \times (f\vec{K}) = f \nabla \times \vec{K} + (\nabla f) \times \vec{K}. \tag{37}$$

Example : Let $\vec{C} = \nabla \times \vec{D}$, then one can easily show $\nabla \cdot \vec{C} = 0$. On the other hand, if one has $\nabla \cdot \vec{C} = 0$, then one can always comes up with a solution for $\vec{C}$ by demanding

$\vec{C} = \nabla \times \vec{D}$ (notice the solution is not unique). In electrodynamics, the magnetic field $\vec{B}$ satisfies $\nabla \cdot \vec{B} = 0$ and receives a solution $\vec{B} = \nabla \times \vec{A}$, here $\vec{A}$ is called vector potential for $\vec{B}$.

Example : Let $\vec{C} = \vec{r}V(r)$, then one finds

$$\nabla \times \vec{C} = \nabla \times (\vec{r}V(r)) = V(r)\nabla \times \vec{r} + (\nabla V(r)) \times \vec{r}. \tag{38}$$

Now using $\nabla \times \vec{r} = 0$ and $\nabla V(r) = \frac{dV(r)}{dr}\vec{r}$, one arrives at

$$\nabla \times (\vec{r}V(r)) = 0. \tag{39}$$

Geometrical interpretation of $\nabla\times$

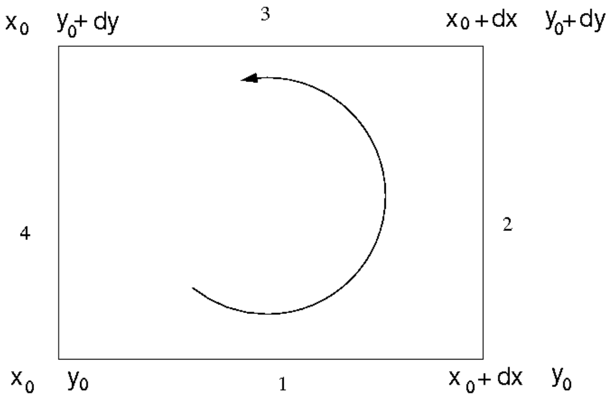

FIG. 8: Circulation around a loop.

Consider following circular line integrals

$$\text{circulation}_{1234} = \int_1 V_x(x,y)d\lambda_x + \int_2 V_y(x,y)d\lambda_y + \int_3 V_x(x,y)d\lambda_x + \int_4 V_y(x,y)d\lambda_y, \tag{40}$$

where segment 1 is from (x_0, y_0) to $(x_0 + dx, y_0)$; segment 2 is from $(x_0 + dx, y_0)$ to $(x_0 + dx, y_0 + dy)$; segment 3 is from $(x_0 + dx, y_0 + dy)$ to $(x_0, y_0 + dy)$; segment 4 is from $(x_0, y_0 + dy)$ to (x_0, y_0). Then using Taylor expansion, one can show

$$
\begin{aligned}
\text{circulation}_{1234} &= V_x(x_0,y_0)dx + \left[V_y(x_0,y_0) + \frac{\partial V_y}{\partial x}|_{x_0,y_0}dx\right]dy \\
&+ \left[V_x(x_0,y_0) + \frac{\partial V_x}{\partial y}|_{x_0,y_0}dy\right](-dx) + V_y(x_0,y_0)(-dy) \\
&= \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)dxdy \qquad (41)
\end{aligned}
$$

Dividing by $dxdy$, one arrives at

$$
\text{Circulation per unit area} = \nabla \times \vec{V}|_z, \qquad (42)
$$

namely the circulation about differential area in the xy-plane is given by the z-component of $\nabla \times \vec{V}$.

A vector field $\vec{B}$ satisfies $\nabla \times \vec{B} = 0$ is called irrotational field. We will come back to this when discussing Maxwell's equations.

Successive applications of ∇ With these introduced gradient, divengence and curl, one can successively apply these oeprators

- $\nabla \cdot \nabla \phi$
- $\nabla \times \nabla \phi$
- $\nabla \nabla \cdot \vec{V}$
- $\nabla \cdot \nabla \times \vec{V}$
- $\nabla \times (\nabla \times \vec{V})$

Example :

- $\nabla \cdot \nabla \times \vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_2 & A_3 \end{vmatrix} = 0$
- $\nabla \times \nabla f = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \end{vmatrix} = 0$

- $\nabla \times (\nabla \times \vec{A}) = \nabla\nabla \cdot \vec{A} - (\nabla \cdot \nabla)\vec{A}$

Example : Electromagnetic wave equation.

In vacuum, Maxwel'sl equations are given by

$$\nabla \cdot \vec{B} = 0, \tag{43}$$

$$\nabla \cdot \vec{E} = 0, \tag{44}$$

$$\nabla \times \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}, \tag{45}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{46}$$

Above $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields respectively. Furhter ϵ_0 and μ_0 are the electric permittivety and magnetic permeability, respectively.

Now by taking time derivative of eq. 45, one has

$$\frac{\partial}{\partial t} \nabla \times \vec{B} \;=\; \nabla \times \frac{\partial \vec{B}}{\partial t} \,,$$

hence

$$\nabla \times (\nabla \times \vec{E}) = -\epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}. \tag{47}$$

An application of $\nabla \times (\nabla \times \vec{A}) = \nabla\nabla \cdot \vec{A} - \nabla \cdot \nabla \vec{A}$ to eq. 47 would lead to

$$\nabla \cdot \nabla \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \tag{48}$$

which is the wave equation for electric field in vacuum.

Next, notice from the identity $\nabla \cdot \nabla \times \vec{A} = 0$, one sees that one can solve eq. 43 by

$$\vec{B} = \nabla \times \vec{A}, \tag{49}$$

here $\vec{A}$ is called the vector potential of the magnetic field $\vec{B}$. Now putiing $\vec{B} = \nabla \times \vec{A}$ into eq. 46, one arrives at $\nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t})$. Further use of the identity $\nabla \times \nabla f = 0$, one finds that $\vec{E} + \frac{\partial \vec{A}}{\partial t}$ can be written as

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \varphi,$$

or

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \varphi. \tag{50}$$

φ is called nonstatic electric potential. You will learn more about Maxwell's equations in your electrodynamics course.

V. VECTOR INTEGRATIONS

Remember that we mentioned earlier vector integrations are done componentwise. In principles, one can define line integrals, surface integrals and volume integrals of vector and scalar (fields). For example, followings defines line integrals of vector fields

$$\begin{aligned}
&\int_c \phi d\vec{r} \\
&\int_c \vec{V} \cdot d\vec{r} \\
&\int_c \vec{V} \times d\vec{r},
\end{aligned} \tag{51}$$

here c is a contour which can be open (starting point and ending point are different) or closed (starting point and ending point are the same). One has similar definitions for surface and volume integrals of vector and scalar fields as well. We use $\int d\vec{\sigma}$ and $\int d\tau$ for surface and volume integration of vector fields. Specifically, the integration over the differetial elements $d\vec{r}$, $d\vec{\sigma}$ and $d\tau$ are given by

$$\begin{aligned}
&\int d\vec{r} = e_1 \int dx_1 + e_2 \int dx_2 + e_3 \int dx_3, \\
&\int d\vec{\sigma} = e_1 \int dx_2 dx_3 + e_2 \int dx_3 dx_1 + e_3 \int dx_1 dx_2, \\
&\int d\tau = \int dx_1 dx_2 dx_3
\end{aligned} \tag{52}$$

.

A. Line integral over vector fields

Example : Let $\vec{A} = (3x^2+6y, -14yz, 20xz^2)$ and c be a contour defined by $c(t) = (t, t^2, t^3)$, then

$$\begin{aligned} \int_{c(0)\text{ to }c(1)} \vec{A}\cdot d\vec{r} &= \int_{c(0)\text{ to }c(1)} \Big[(3x^2+6y)dx - 14yzdy + 20xz^2dz\Big] \\ &= \int_0^1 \Big[(3t^2+6t^2)dt - 14t^2(2tdt) + (20t)(t^3)(3t^2dt)\Big] \\ &= 5. \end{aligned} \tag{53}$$

Question : Let $\vec{F} = (3xy, -y^2, 0)$ and let c be the curve defined by $y = 2x^2$, what is $\int_c \vec{F}\cdot d\vec{r}$ from $(0,0)$ to $(1,2)$?

Solution :

$$\int_c \vec{F}\cdot d\vec{r} = \int_0^1 \Big[(3x(2x^2)dx - (2x^2)^2 4xdx\Big] = -\frac{7}{6}. \tag{54}$$

B. Surface integral over vecor fields

Question : Let $\vec{A} = (18z, -12, 3y)$ and S be a surface defined by $2x+3y+6z = 12$. What's $\int_S \vec{A}\cdot\vec{n}dS$ over the area with $x \geq 0$, $y \geq 0$, $z \geq 0$?

Hits : Remember ∇S is perpeticular to S. Hence the normal vector $\vec{n}$ is given by $\nabla S/|\nabla S| = (2/7, 3/7, 6/7)$. Also $dS = dxdy/|\vec{n}\cdot\vec{k}|$. Finally using $z = \frac{12-2x-3y}{6}$, one has

$$\int_S \vec{A}\cdot\vec{n}dS = \int_R (6-2x)dxdy, \tag{55}$$

here R is the projection area of S onto the $x-y$ plane.

C. Integral definitions of Gradient, Divergence and Curl

The operations Gradient, Divergence and Curl defined earlier can be defined through vector integration :

$$\begin{aligned}
\nabla\phi &= \lim_{\int d\tau\to 0}\frac{\int \phi d\vec{\sigma}}{\int d\tau} \\
\nabla\cdot\vec{V} &= \lim_{\int d\tau\to 0}\frac{\int \vec{V}\cdot d\vec{\sigma}}{\int d\tau} \\
\nabla\times\vec{V} &= \lim_{\int d\tau\to 0}\frac{\int d\vec{\sigma}\times\nabla\vec{V}}{\int d\tau}
\end{aligned} \tag{56}$$

Let's quickly give a proof for the first equation in eq. 56. Consider a 3-dim rectangular box, one has

$$\begin{aligned}
\int \phi d\vec{\sigma} &= -e_x\int\left[\phi-\frac{\partial\phi}{\partial x}\frac{dx}{2}\right]dydz + e_x\int\left[\phi+\frac{\partial\phi}{\partial x}\frac{dx}{2}\right]dydz \\
&\quad -e_y\int\left[\phi-\frac{\partial\phi}{\partial y}\frac{dy}{2}\right]dxdz + e_y\int\left[\phi+\frac{\partial\phi}{\partial y}\frac{dy}{2}\right]dxdz \\
&\quad -e_z\int\left[\phi-\frac{\partial\phi}{\partial z}\frac{dz}{2}\right]dxdy + e_z\int\left[\phi+\frac{\partial\phi}{\partial z}\frac{dz}{2}\right]dxdy \\
&= \int\left[e_x\frac{\partial\phi}{\partial x}+e_y\frac{\partial\phi}{\partial y}+e_z\frac{\partial\phi}{\partial z}\right]dxdydz.
\end{aligned} \tag{57}$$

Dividing above equation by $\int d\tau = \int dxdydz$, we prove the frist equation of eq. 56.

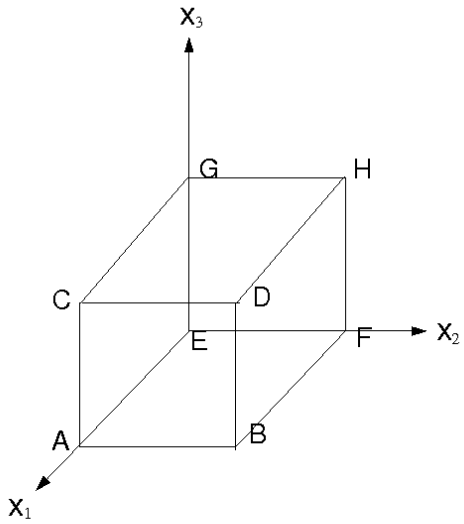

FIG. 9: rectangular parallelepiped.

D. Stoke's Theorem

Let S be a closed surface and l be its boundary, then Stoke's theorem says

$$\oint_l \vec{V} \cdot d\vec{r} = \int_S \nabla \times \vec{V} \cdot d\vec{\sigma} \tag{58}$$

provided that $\vec{V}$ has continuous derivatives inside S. To prove Stoke's theorem, let's subdivide the surface into very small rectangles. Then by applying the same arguments in deriving eq. 60, one has

$$\sum_{\text{four sides}} \vec{V} \cdot d\vec{r} = \nabla \times \vec{V} \cdot d\vec{\sigma}. \tag{59}$$

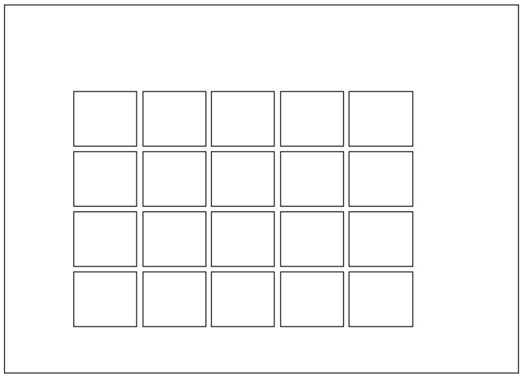

FIG. 10: Proof of Stoke's Theorem.

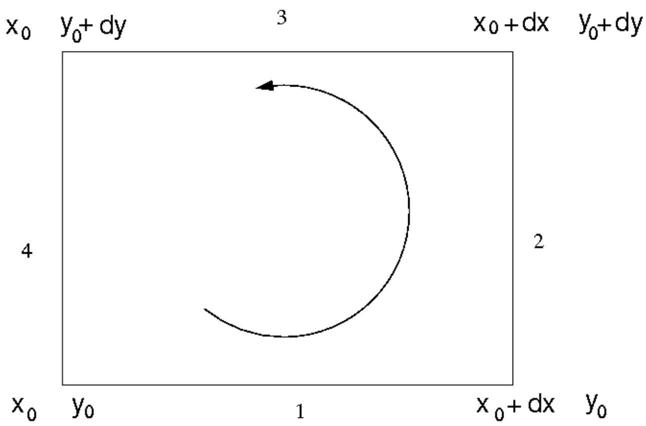

FIG. 11: Circulation around a loop.

$$\text{circulation}_{1234} = \int_1 V_x(x,y)d\lambda_x + \int_2 V_y(x,y)d\lambda_y + \int_3 V_x(x,y)d\lambda_x + \int_4 V_y(x,y)d\lambda_y, \tag{60}$$

$$
\begin{aligned}
\text{circulation}_{1234} &= V_x(x_0,y_0)dx + \left[V_y(x_0,y_0) + \frac{\partial V_y}{\partial x}|_{x_0,y_0}dx\right]dy \\
&+ \left[V_x(x_0,y_0) + \frac{\partial V_x}{\partial y}|_{x_0,y_0}dy\right](-dx) + V_y(x_0,y_0)(-dy) \\
&= \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)dxdy \qquad (61)
\end{aligned}
$$

Further we notice that all the interior line segments inside S cancel identically. Hence when we sum over all rectangles, we reach

$$\sum_{\text{exterior line segments}} \vec{V}\cdot d\vec{r} = \sum_{\text{rectangles}} \nabla\times\vec{V}\cdot d\vec{\sigma}. \qquad (62)$$

By taking the limit of making the number of rectangles to infinity , we arrive at

$$\oint_l \vec{V}\cdot d\vec{r} = \int_S \nabla\times\vec{V}\cdot d\vec{\sigma}. \qquad (63)$$

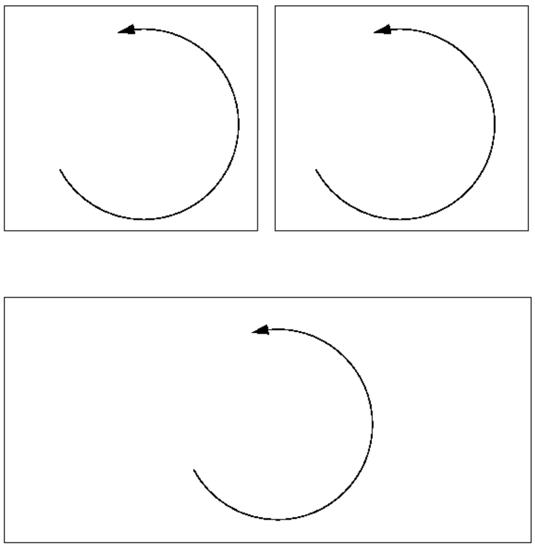

FIG. 12: Cancelation on interior paths.

E. Gauss's Theorem and Green's Theorem

Similarly, using the same cosideration as we did in proving Stoke's theorem, one can prove Gauss's theorem

$$\int_S \vec{V}\cdot d\vec{\sigma} = \int_V \nabla\cdot\vec{V}d\tau, \qquad (64)$$

here V is a volume and S is the corresponding surface of V.

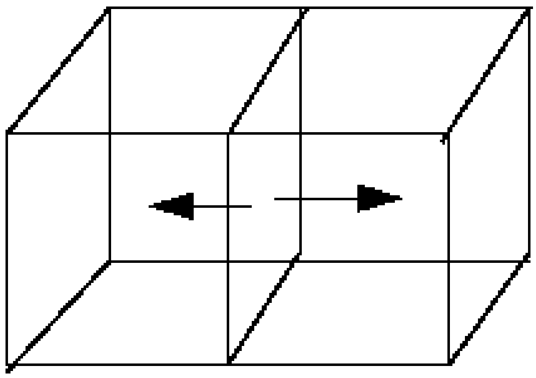

FIG. 13: Cancelation on interior surfaces.

By applying Gauss's theorem to the following identity

$$\nabla \cdot (u\nabla v) = u\nabla u \cdot \nabla v + (\nabla u) \cdot (\nabla v), \tag{65}$$

we obtain Green's thoerem

$$\int_S u\nabla v \cdot d\vec{\sigma} = \int_V u\nabla \cdot \nabla v d\tau + \int_V \nabla u \cdot \nabla v d\tau. \tag{66}$$

Example : Oersted's and Faraday's Laws.

Consider the magnetic field generated by a long wire that carries a stationary current I (remember $\vec{B}$ is along the wire and $\vec{E}$ is perpeticular to $\vec{B}$). From Maxwell's equation $\nabla \times \vec{H} = \vec{J}$, by applying Stoke's theorem to (integrating) a area S perpenticular to the wire, one reaches the following Oersteds' law

$$I = \int_S \vec{J} \cdot d\vec{\sigma} = \int_S (\nabla \times \vec{H}) \cdot d\vec{\sigma} = \oint_{\partial S} \vec{H} \cdot d\vec{r}. \tag{67}$$

Applying similar consideration to the Maxwell's equation $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and employing the Stoke's theorem, we arrive at the Faraday's law

$$\int_{\partial S} \vec{E} \cdot d\vec{r} = \int_S (\nabla \times \vec{E}) \cdot d\vec{\sigma} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{\sigma} = -\frac{d\Phi}{dt}, \tag{68}$$

VI. POTENTIAL THEORY

A. Scalar Potential

If a force $\vec{F}$ can be written as $\vec{F} = -\nabla\varphi$ for some scalar function φ in a simple connected area (no holes inside the area), then one has

$$\nabla \times \vec{F} = \nabla \times (\nabla\varphi) = 0 \tag{69}$$

Now using Stoke's theorem, one further finds

$$\int_S \nabla \times \vec{F} \cdot d\sigma = \oint_C \vec{F} \cdot d\vec{r} = 0, \tag{70}$$

here S is any simple connected area within the large area and C is the oriented boundary of S. Above result implies that for any 2 points A and B inside the area, the line integral over $\vec{F}$ along any simple connected curve connecting A and B are the same, namely the integral is independent of path connecting A and B. Such force $\vec{F}$ which can be written as the gradient of a single-value scalar function (potential) φ is called a conservative force. Actually one can show that a single-value scalar potential φ for a force $\vec{F}$ exists if and only if $\nabla \times \vec{F} = 0$ or the work done around every closed loop is zero.

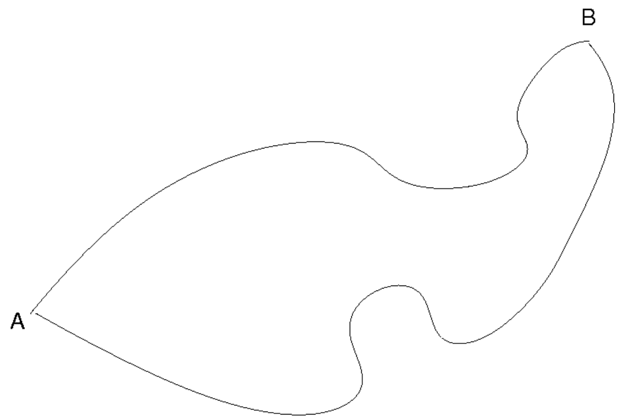

FIG. 14: Possible paths for doing work.

Exampe : for the force of simple harmonic oscillation $\vec{F} = -k\vec{r}$, the corresponding scalar potential φ is given by

$$\varphi(r) = \int_0^r d\varphi = \int_0^r \nabla\varphi \cdot d\vec{r} = -\int_0^r \vec{F} \cdot d\vec{r} = -\frac{1}{2}kr^2. \tag{71}$$

B. Vector Potential

Earlier we introduced a vector potential $\vec{A}$ such that $\vec{B}$ is given by $\vec{B} = \nabla \times \vec{A}$. Hence one immediately has $\nabla \cdot \vec{B} = 0$. Actually for a given $\vec{B}$ such that $\nabla \cdot \vec{B} = 0$, then one can show a vector potential $\vec{A}$ exits (not unique!).

Example : Let $\vec{B} = (0, 0, B_z)$, then one finds that a vector potential $\vec{A}$ for $\vec{B}$ can be given by $\vec{A} = (0, xB_z, 0)$. Notice one can show that $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$ is another vector potential for $\vec{B}$ (hence indeed $\vec{A}$ is not unique). You will learn more about these in your electrodynamics class.
