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Final Exam of Differential Geometry I

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1. Compute $\kappa, \tau, \overline{T}, \overline{N}, \overline{B}$ for the space curve $\vec{\gamma}$

$$\vec{\gamma}(t) = \left(\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1-t)^{\frac{3}{2}}, \frac{t}{\sqrt{2}}\right).$$

解：$\|\vec{\gamma}'(t)\| = \sqrt{\frac{1}{4}(1+t)+\frac{1}{4}(1-t)+\frac{1}{2}} = 1 \Rightarrow t$ 是 $\vec{\gamma}$ 的arclength parameter

$$\overline{T}(t) = \vec{\gamma}'(t) = \left(\frac{1}{2}(1+t)^{\frac{1}{2}}, -\frac{1}{2}(1-t)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\overline{K}(t) = \overline{T}'(t) = \left(\frac{1}{4}(1+t)^{-\frac{1}{2}}, \frac{1}{4}(1-t)^{-\frac{1}{2}}, 0\right)$$

$$\kappa = \|\overline{T}'(t)\| = \frac{1}{4}\sqrt{\frac{1}{1+t}+\frac{1}{1-t}} = \frac{1}{4}\sqrt{\frac{2}{1-t^2}} = \frac{1}{\sqrt{8(1-t^2)}}$$

$$\overline{N}(t) = \frac{\overline{K}(t)}{\kappa} = \sqrt{8(1-t^2)}\left(\frac{1}{4}(1+t)^{-\frac{1}{2}}, \frac{1}{4}(1-t)^{-\frac{1}{2}}, 0\right) = \frac{1}{\sqrt{2}}\left((1-t)^{\frac{1}{2}}, (1+t)^{\frac{1}{2}}, 0\right)$$

$$\overline{N}'(t) = \frac{1}{2\sqrt{2}}\left(-(1-t)^{-\frac{1}{2}}, (1+t)^{-\frac{1}{2}}, 0\right)$$

$$\overline{B}(t) = \overline{T}(t)\times\overline{N}(t) = \left(-\frac{1}{2}(1+t)^{\frac{1}{2}}, \frac{1}{2}(1-t)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\tau = <\overline{N}'(t), \overline{B}(t)> = \frac{1}{4\sqrt{2}}\left(\sqrt{\frac{1+t}{1-t}}+\sqrt{\frac{1-t}{1+t}}\right) = \frac{1}{\sqrt{8(1-t^2)}}$$

2.Let the curve $\vec{\gamma}$ be

$$\vec{\gamma}(t) = \left(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t\right).$$

Show that the curvature $\vec{\gamma}$ is a circle, and find the centre, radius, and the plane it lies.

解：$\|\vec{\gamma}'(t)\| = \sqrt{\frac{16}{25}\sin^2 t+\cos^2 t+\frac{9}{25}\sin^2 t} = 1 \Rightarrow t$ 是 $\vec{\gamma}$ 的arclength parameter

$$\overline{T}(t) = \vec{\gamma}'(t) = \left(-\frac{4}{5}\sin t, -\cos t, \frac{3}{5}\sin t\right)$$

$$\overline{K}(t) = \overline{T}'(t) = \left(-\frac{4}{5}\cos t, \sin t, \frac{3}{5}\cos t\right)$$

$\kappa = \left\|\overline{T}\,'(t)\right\| = \sqrt{\frac{16}{25}\cos^2 t + \sin^2 t + \frac{9}{25}\cos^2 t} = 1, \therefore \kappa$ 是constant

$\Rightarrow \overline{N}(t) = \overline{K}(t)$

$\overline{N}\,'(t) = \left(\frac{4}{5}\sin t, \cos t, -\frac{3}{5}\sin t\right)$

$\overline{B}(t) = \overline{T}(t) \times \overline{N}(t) = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right)$

$\tau = <\overline{N}\,'(t), \overline{B}(t)> = 0$

再加上 $\vec{\gamma}$ 是continuous function, 且 $\vec{\gamma}(0) = \vec{\gamma}(2\pi) \therefore \vec{\gamma}$ 是一個circle

其radius為 $\frac{1}{\kappa} = 1$ 它的centre為 $\vec{\gamma}(t) + \frac{1}{\kappa}\overline{N}(t) = (0,1,0)$

此plane過 $(0,1,0)$ 且與 $\overline{B}(t) = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right)$ perpendicular

故此plane為 $3x + 4z = 0$

3.Let Enneper's surface be described by the parametrized surface,

$$\vec{x}(u,v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2).$$

(a)What is the Gaussian curvature K at the point $(x,y,z) = (0,0,0)$

(b)What is the mean curvature H at the point $(x,y,z) = (0,0,0)$?

(c)What is the normal curvature κ_n at the point $(x,y,z) = (0,0,0)$ along a horizontal direction?

解： $\overline{x_u}(u,v) = (1 - u^2 + v^2, 2vu, 2u)$

$\overline{x_v}(u,v) = (2uv, 1 - v^2 + u^2, -2v)$

$E(u,v) = \left\|\overline{x_u}(u,v)\right\|^2 = (1 + u^2 + v^2)^2, F(u,v) = <\overline{x_u}(u,v), \overline{x_u}(u,v)> = 0$

$G(u,v) = \left\|\overline{x_v}(u,v)\right\|^2 = (1 + u^2 + v^2)^2$

$\mathcal{F}_I(u,v) = \begin{bmatrix} (1+u^2+v^2)^2 & 0 \\ 0 & (1+u^2+v^2)^2 \end{bmatrix}$

$\overline{x_u}(u,v) \times \overline{x_v}(u,v) = (-2u(1+u^2+v^2), 2v(1+u^2+v^2), (1+u^2+v^2)(1-u^2-v^2))$

$\left\|\overline{x_u}(u,v) \times \overline{x_v}(u,v)\right\| = (1+u^2+v^2)^2$

$\overline{U}(u,v) = \left(\frac{-2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2}\right)$

$\overline{x_{uu}}(u,v) = (-2u, 2v, 2), \overline{x_{uv}}(u,v) = (2v, 2u, 0), \overline{x_{vv}}(u,v) = (2u, -2v, -2)$

$l(u,v) = <\overline{U}(u,v), \overline{x_{uu}}(u,v)> = 2, m = <\overline{U}(u,v), \overline{x_{uv}}(u,v)> = 0$

$n(u,v) = <\overline{U}(u,v), \overline{x_{vv}}(u,v)> = -2$

$$\mathcal{F}_{II}(u,v)=\begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix}$$

$$\mathcal{F}_I^{-1}(u,v)\mathcal{F}_{II}(u,v)=\frac{2}{(1+u^2+v^2)^2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$\vec{x}(0,0)=(0,0,0)$$

(a) $K(u,v)=\det(F_I^{-1}(u,v)F_{II}(u,v))=\dfrac{-4}{(1+u^2+v^2)^4}$

$K(0,0)=-4$

(b) $H(u,v)=\dfrac{1}{2}tr(F_I^{-1}(u,v)F_{II}(u,v))=0$

$H(0,0)=0$

(c) $u^2-v^2=0$

$(u+v)(u-v)=0$

$\therefore u=\pm v$

① $u=v$

令 $\vec{\gamma}(u)=\vec{x}(u,u)=\left(u+\dfrac{2}{3}u^3,u+\dfrac{2}{3}u^3,0\right)$

$\vec{\gamma}\,'(u)=(1+2u^2,1+2u^2,0),s(u)=\left\|\vec{\gamma}\,'(u)\right\|=\sqrt{2}(1+2u^2)$

$$\frac{\mathrm{d}\vec{\gamma}}{\mathrm{d}s}=\frac{\dfrac{\mathrm{d}\vec{\gamma}}{\mathrm{d}u}}{\dfrac{\mathrm{d}s}{\mathrm{d}u}}=\frac{(1+2u^2,1+2u^2,0)}{4\sqrt{2}u}$$

$$\frac{\mathrm{d}^2\vec{\gamma}}{\mathrm{d}s^2}=\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\vec{\gamma}}{\mathrm{d}s}=\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{(1+2u^2,1+2u^2,0)}{4\sqrt{2}u}\right]=\frac{\dfrac{\mathrm{d}}{\mathrm{d}u}\left[\dfrac{(1+2u^2,1+2u^2,0)}{4\sqrt{2}u}\right]}{\dfrac{\mathrm{d}s}{\mathrm{d}u}}$$

$$=\frac{\dfrac{4\sqrt{2}u(4u,4u,0)-(1+2u^2,1+2u^2,0)\cdot 4\sqrt{2}}{32u^2}}{4\sqrt{2}u}$$

$$=\frac{(-2u^2+4u-1,-2u^2+4u-1,0)}{32u^3}$$

$\because \dfrac{\mathrm{d}^2\vec{\gamma}}{\mathrm{d}s^2}(0,0)$ 不存在，因此normal curvature不存在

② $u=-v$

令 $\vec{\gamma}(u)=\vec{x}(u,-u)=\left(u+\dfrac{2}{3}u^3,-u-\dfrac{2}{3}u^3,0\right)$

$\vec{\gamma}\,'(u)=(1+2u^2,-1-2u^2,0),s(u)=\left\|\vec{\gamma}\,'(u)\right\|=\sqrt{2}(1+2u^2)$

$$\frac{d\vec{\gamma}}{ds}=\frac{\frac{d\vec{\gamma}}{du}}{\frac{ds}{du}}=\frac{(1+2u^2,-1-2u^2,0)}{4\sqrt{2}u}$$

$$\frac{d^2\vec{\gamma}}{ds^2}=\frac{d}{ds}\frac{d\vec{\gamma}}{ds}=\frac{d}{ds}\left[\frac{(1+2u^2,-1-2u^2,0)}{4\sqrt{2}u}\right]=\frac{\frac{d}{du}\left[\frac{(1+2u^2,-1-2u^2,0)}{4\sqrt{2}u}\right]}{\frac{ds}{du}}$$

$$=\frac{\frac{4\sqrt{2}u(4u,-4u,0)-(1+2u^2,-1-2u^2,0)\cdot 4\sqrt{2}}{32u^2}}{4\sqrt{2}u}$$

$$=\frac{(-2u^2+4u-1,2u^2-4u+1,0)}{32u^3}$$

$\because \frac{d^2\vec{\gamma}}{ds^2}(0,0)$ 不存在，因此normal curvature不存在

4.Let $z^2=x^2+y^2-1$ be a hyperboloid of one-sheet in $\mathbb{R}^3$.

(a)What is the first fundamental form?

(b)What is the second fundamental form?

(c)What is the principal curvatures κ_1 and κ_2?

(d)Compute the area of the image from $\{(x,y):1\le x^2+y^2\le 4\}$ under the Gauss map.

解：令 $\vec{x}(u,v)=(\cosh u\cos v,\cosh u\sin v,\sinh u),u\in\mathbb{R},-\pi<v\le\pi$

$\overline{x_u}(u,v)=(\sinh u\cos v,\sinh u\sin v,\cosh u)$

$\overline{x_v}(u,v)=(-\cosh u\sin v,\cosh u\cos v,0)$

$\overline{x_u}(u,v)\times\overline{x_v}(u,v)=(-\cosh^2 u\cos v,-\cosh^2 u\sin v,\cosh u\sinh u)$

$\left\|\overline{x_u}(u,v)\times\overline{x_v}(u,v)\right\|=\cosh u\sqrt{\cosh^2 u+\sinh^2 u}$

$$\vec{U}(u,v)=\frac{(-\cosh u\cos v,-\cosh u\sin v,\sinh u)}{\sqrt{\cosh^2 u+\sinh^2 u}}$$

$\overline{x_{uu}}(u,v)=(\cosh u\cos v,\cosh u\sin v,\sinh u)$

$\overline{x_{uv}}(u,v)=(-\sinh u\sin v,\sinh u\cos v,0)$

$\overline{x_{vv}}(u,v)=(-\cosh u\cos v,-\cosh u\sin v,0)$

$E(u,v)=\left\|\overline{x_u}(u,v)\right\|^2=\sinh^2 u+\cosh^2 u, F(u,v)=<\overline{x_u}(u,v),\overline{x_v}(u,v)>=0$

$G(u,v)=\left\|\overline{x_v}(u,v)\right\|^2=\cosh^2 u$

$$l(u,v)=<\vec{U}(u,v),\overline{x_{uu}}(u,v)>=-\frac{1}{\sqrt{\cosh^2 u+\sinh^2 u}},m=<\vec{U}(u,v),\overline{x_{uv}}(u,v)>=0$$

$$n(u,v)=<\overline{U}(u,v),\overrightarrow{x_{vv}}(u,v)>=\frac{\cosh^2 u}{\sqrt{\cosh^2 u+\sinh^2 u}}$$

(a) $\mathcal{F}_I(u,v)=\begin{bmatrix}\sinh^2 u+\cosh^2 u & 0\\ 0 & \cosh^2 u\end{bmatrix}$

(b) $\mathcal{F}_{II}(u,v)=\dfrac{1}{\sqrt{\cosh^2 u+\sinh^2 u}}\begin{bmatrix}-1 & 0\\ 0 & \cosh^2 u\end{bmatrix}$

(c) $\mathcal{F}_I^{-1}(u,v)\mathcal{F}_{II}(u,v)=\dfrac{1}{\sqrt{\cosh^2 u+\sinh^2 u}}\begin{bmatrix}-\dfrac{1}{\sinh^2 u+\cosh^2 u} & 0\\ 0 & 1\end{bmatrix}$

$$\therefore \kappa_1=\frac{1}{\sqrt{\cosh^2 u+\sin^2 u}},\kappa_2=-\frac{1}{\sqrt{(\cosh^2 u+\sin^2 u)^3}}$$

(d) $K(u,v)=\det(\mathcal{F}_I^{-1}(u,v)\mathcal{F}_{II}(u,v))=-\dfrac{1}{(\cosh^2 u+\sinh^2 u)^2}$

$$x^2+y^2=\cosh^2 u=\left(\frac{e^u+e^{-u}}{2}\right)=\frac{e^{2u}+2+e^{-2u}}{4}$$

$$\frac{e^{2u}+2+e^{-2u}}{4}=\frac{(e^{2u}+e^{-2u})+2}{4}\geq\frac{2+2}{4}=1,\forall u\in\mathbb{R}$$

$$\frac{e^{2u}+2+e^{-2u}}{4}\leq 4$$

$$e^{2u}+2+e^{-2u}\leq 16$$

$$e^{2u}-14+e^{-2u}\leq 0$$

$$e^{4u}-14e^{2u}+1\leq 0$$

$$7-2\sqrt{12}\leq e^{2u}\leq 7+2\sqrt{12}$$

$$4-\sqrt{3}\leq e^{u}\leq 4+\sqrt{3}$$

$$\ln(4-\sqrt{3})\leq u\leq \ln(4+\sqrt{3})$$

$$\begin{aligned}\text{area}&=\int_{\ln(4-\sqrt{3})}^{\ln(4+\sqrt{3})}\int_{-\pi}^{\pi}\left\|\overline{U}_u(u,v)\times\overline{U}_v(u,v)\right\|\mathrm{d}v\mathrm{d}u\\
&=\int_{\ln(4-\sqrt{3})}^{\ln(4+\sqrt{3})}\int_{-\pi}^{\pi}\left|K(u,v)\right|\left\|\overrightarrow{x_u}(u,v)\times\overrightarrow{x_v}(u,v)\right\|\mathrm{d}v\mathrm{d}u\\
&=\int_{\ln(4-\sqrt{3})}^{\ln(4+\sqrt{3})}\int_{-\pi}^{\pi}\frac{1}{(\cosh^2 u+\sinh^2 u)^2}\cdot\cosh u\sqrt{\cosh^2 u+\sinh^2 u}\,\mathrm{d}v\mathrm{d}u\\
&=\int_{\ln(4-\sqrt{3})}^{\ln(4+\sqrt{3})}\int_{-\pi}^{\pi}\frac{\cosh u}{\sqrt{(\cosh^2 u+\sinh^2 u)^3}}\mathrm{d}v\mathrm{d}u\\
&=2\pi\int_{\ln(4-\sqrt{3})}^{\ln(4+\sqrt{3})}\frac{\cosh u}{\sqrt{(1+2\sinh^2 u)^3}}\mathrm{d}u\end{aligned}$$

(令 $x=\sinh u,\mathrm{d}x=\cosh u\mathrm{d}u$

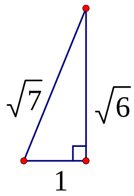
$$1 \le \cosh^2 u \le 4 \Rightarrow 1 \le \sinh^2 u + 1 \le 4, \sinh^2 u \le 3 \Rightarrow -\sqrt{3} \le \sinh u \le \sqrt{3})$$

$$= 2\pi \int_{-\sqrt{3}}^{\sqrt{3}} \frac{\mathrm{d}x}{\sqrt{(1+2x^2)^3}}$$

$$(\text{令 } x = \frac{1}{\sqrt{2}} \tan\theta, \mathrm{d}x = \frac{1}{\sqrt{2}} \sec^2\theta \mathrm{d}\theta$$

$$x = \sqrt{3} \to \theta = \tan^{-1}\sqrt{6}, x = -\sqrt{3} \to \theta = -\tan^{-1}\sqrt{6})$$

$$= 2\pi \int_{-\tan^{-1}\sqrt{6}}^{\tan^{-1}\sqrt{6}} \frac{1}{\sec^3\theta} \cdot \frac{1}{\sqrt{2}} \sec^2\theta \mathrm{d}\theta$$



$$= \sqrt{2}\pi \int_{-\tan^{-1}\sqrt{6}}^{\tan^{-1}\sqrt{6}} \frac{1}{\sec\theta} \mathrm{d}\theta$$

$$= 2\sqrt{2}\pi \int_0^{\tan^{-1}\sqrt{6}} \cos\theta \mathrm{d}\theta = 2\sqrt{2}\pi \sin\theta \Big|_0^{\tan^{-1}\sqrt{6}} = 2\sqrt{2}\pi \cdot \sqrt{\frac{6}{7}} = 4\sqrt{\frac{3}{7}}\pi = \frac{4}{7}\sqrt{21}\pi$$