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Final Exam of Differential Geometry I

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1. Let the curve $\overline{\beta}(s)=\left(\frac{1}{\sqrt{2}}\cos s,\sin s,\frac{1}{\sqrt{2}}\cos s\right)$. Find the curvature κ and the torsion τ of $\overline{\beta}$.

解：$\left\|\overline{\beta}'(s)\right\|=\sqrt{\frac{1}{2}\sin^2 s+\cos^2 s+\frac{1}{2}\sin^2 s}=1\Rightarrow s$ 是 $\overline{\beta}$ 的arclength parameter

$$\overline{T}(s)=\overline{\beta}'(s)=\left(-\frac{1}{\sqrt{2}}\sin s,\cos s,-\frac{1}{\sqrt{2}}\sin s\right)$$

$$\overline{K}(s)=\overline{T}'(s)=\left(-\frac{1}{\sqrt{2}}\cos s,-\sin s,-\frac{1}{\sqrt{2}}\cos s\right)$$

$$\kappa=\left\|\overline{T}'(s)\right\|=\sqrt{\frac{1}{2}\cos^2 s+\sin^2 s+\frac{1}{2}\cos^2 s}=1\Rightarrow \overline{N}(s)=\overline{K}(s)$$

$$\overline{N}'(s)=\left(\frac{1}{\sqrt{2}}\sin s,-\cos s,\frac{1}{\sqrt{2}}\sin s\right)$$

$$\overline{B}(s)=\overline{T}(s)\times\overline{N}(s)=\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$$

$$\tau=<\overline{N}'(s),\overline{B}(s)>=0$$

2. Let a plane curve be defined by polar coordinates, $\rho=\rho(\theta)$, where $\rho>0$ and $\theta\in[0,2\pi)$.

(a)Show that the arclength is

$$\int_0^{2\pi}\sqrt{\rho^2+(\rho')^2}\,\mathrm{d}\theta.$$

(b)Show that the curvature is

$$\kappa(\theta)=\frac{2(\rho')^2-\rho\rho''+\rho^2}{\sqrt{\left[(\rho')^2+\rho^2\right]^3}}$$

解：令此curve $\vec{\gamma}(\theta)=(\rho(\theta)\cos\theta,\rho(\theta)\sin\theta,0)$

(a) $s=\int_0^{2\pi}\left\|\vec{\gamma}'(\theta)\right\|\mathrm{d}\theta$

$$=\int_0^{2\pi}\sqrt{(\rho'\cos\theta-\rho\sin\theta)^2+(\rho'\sin\theta+\rho\cos\theta)^2}\,\mathrm{d}\theta=\int_0^{2\pi}\sqrt{\rho^2+(\rho')^2}\,\mathrm{d}\theta$$

(b) $\overline{\gamma}'(\theta)=(\rho'\cos\theta-\rho\sin\theta,\rho'\sin\theta+\rho\cos\theta,0)$

$$\overrightarrow{\gamma}''(\theta)=(\rho''\cos\theta-2\rho'\sin\theta-\rho\cos\theta,\rho''\sin\theta+2\rho'\cos\theta-\rho\sin\theta,0)$$
$$\overrightarrow{\gamma}'(\theta)\times\overrightarrow{\gamma}''(\theta)=(0,0,2(\rho')^2-\rho\rho''+\rho^2)$$
$$\left\|\overrightarrow{\gamma}'(\theta)\right\|^3=\sqrt{\left[\rho^2+(\rho')^2\right]^3}$$
$$\therefore\kappa=\frac{\left\|\overrightarrow{\gamma}'(\theta)\times\overrightarrow{\gamma}''(\theta)\right\|}{\left\|\overrightarrow{\gamma}'(\theta)\right\|^3}=\frac{2(\rho')^2-\rho\rho''+\rho^2}{\sqrt{\left[\rho^2+(\rho')^2\right]^3}}$$

3.Let $\overrightarrow{\alpha}(t)$ be a regular curve in $\mathbb{R}^3$. Prove the followings:

(a)The curvature

$$\kappa=\frac{\left\|\overrightarrow{\alpha}'(t)\times\overrightarrow{\alpha}''(t)\right\|}{\left\|\overrightarrow{\alpha}'(t)\right\|^3}$$

(b)The torsion

$$\tau=\frac{<\overrightarrow{\alpha}'(t)\times\overrightarrow{\alpha}''(t),\overrightarrow{\alpha}'''(t)>}{\left\|\overrightarrow{\alpha}'(t)\times\overrightarrow{\alpha}''(t)\right\|^2}$$

證：(a)令 s 是 $\overrightarrow{\alpha}$ 的arclength parameter, $v(t)=\dfrac{\mathrm{d}s(t)}{\mathrm{d}t}$

$$\overrightarrow{\alpha}'(t)=\overrightarrow{\alpha}'(s)\frac{\mathrm{d}s}{\mathrm{d}t}=v(t)\overrightarrow{T}(s)$$
$$\overrightarrow{\alpha}''(t)=v'(t)\overrightarrow{T}(s)+v(t)\overrightarrow{T}'(s)\frac{\mathrm{d}s}{\mathrm{d}t}=v'(t)\overrightarrow{T}(s)+[v(t)]^2\kappa(s)\overrightarrow{N}(s)$$
$$\overrightarrow{\alpha}'(t)\times\overrightarrow{\alpha}''(t)=[v(t)]^3\kappa(s)\overrightarrow{B}(s)$$
$$\frac{\left\|\overrightarrow{\alpha}'(t)\times\overrightarrow{\alpha}''(t)\right\|}{\left\|\overrightarrow{\alpha}'(t)\right\|^3}=\frac{[v(t)]^3\kappa(s)}{[v(t)]^3}=\kappa(s)$$

(b) $$\overrightarrow{\alpha}'''(t)=v''(t)\overrightarrow{T}(s)+v'(t)\overrightarrow{T}'(s)\frac{\mathrm{d}s}{\mathrm{d}t}+\left\{2v(t)v'(t)\kappa(s)+[v(t)]^2\kappa'(s)\frac{\mathrm{d}s}{\mathrm{d}t}\right\}\overrightarrow{N}(s)$$
$$+[v(t)]^2\kappa(s)\overrightarrow{N}'(s)\frac{\mathrm{d}s}{\mathrm{d}t}$$
$$=\left\{v''(t)-[v(t)]^3[\kappa(s)]^2\right\}\overrightarrow{T}(s)$$
$$+\left\{v(t)v'(t)\kappa(s)+2v(t)v'(t)\kappa(s)+[v(t)]^3\kappa'(s)\right\}\overrightarrow{N}(s)$$
$$+[v(t)]^3\kappa(s)\tau(s)\overrightarrow{B}(s)$$
$$<\overrightarrow{\alpha}'(t)\times\overrightarrow{\alpha}''(t),\overrightarrow{\alpha}'''(t)>=[v(t)]^6[\kappa(s)]^2\tau(s)$$
$$\frac{<\overrightarrow{\alpha}'(t)\times\overrightarrow{\alpha}''(t),\overrightarrow{\alpha}'''(t)>}{\left\|\overrightarrow{\alpha}'(t)\times\overrightarrow{\alpha}''(t)\right\|^2}=\frac{[v(t)]^6[\kappa(s)]^2\tau(s)}{[v(t)]^6[\kappa(s)]^2}=\tau(s)$$

4.Let $z^2=x^2+y^2-1$ be a hyperboloid of one-sheet in $\mathbb{R}^3$.

(a)What is the Gaussian curvature K at the point $(x,y,z)=(1,0,0)$?

(b)What is the mean curvature H at the point $(x,y,z)=(1,0,0)$?

(c)What is the normal curvature κ_n at the point $(x,y,z)=(1,0,0)$ along a horizontal direction?

Hint: To simplifying calculation, you might need to find a proper parametrization of the hyperboloid of one-sheet by using the functions of sinh and cosh.

解：令 $\vec{x}(u,v)=(\cosh u\cos v,\cosh u\sin v,\sinh u), u\in\mathbb{R}, -\pi<v\le\pi$

$\overrightarrow{x_u}(u,v)=(\sinh u\cos v,\sinh u\sin v,\cosh u)$

$\overrightarrow{x_v}(u,v)=(-\cosh u\sin v,\cosh u\cos v,0)$

$\overrightarrow{x_u}(u,v)\times\overrightarrow{x_v}(u,v)=(-\cosh^2 u\cos v,-\cosh^2 u\sin v,\cosh u\sinh u)$

$\left\|\overrightarrow{x_u}(u,v)\times\overrightarrow{x_v}(u,v)\right\|=\cosh u\sqrt{\cosh^2 u+\sinh^2 u}$

$$\vec{U}(u,v)=\frac{(-\cosh u\cos v,-\cosh u\sin v,\sinh u)}{\sqrt{\cosh^2 u+\sinh^2 u}}$$

$\overrightarrow{x_{uu}}(u,v)=(\cosh u\cos v,\cosh u\sin v,\sinh u)$

$\overrightarrow{x_{uv}}(u,v)=(-\sinh u\sin v,\sinh u\cos v,0)$

$\overrightarrow{x_{vv}}(u,v)=(-\cosh u\cos v,-\cosh u\sin v,0)$

$E(u,v)=\left\|\overrightarrow{x_u}(u,v)\right\|^2=\sinh^2 u+\cosh^2 u, F(u,v)=<\overrightarrow{x_u}(u,v),\overrightarrow{x_v}(u,v)>=0$

$G(u,v)=\left\|\overrightarrow{x_v}(u,v)\right\|^2=\cosh^2 u$

$$l(u,v)=<\vec{U}(u,v),\overrightarrow{x_{uu}}(u,v)>=-\frac{1}{\sqrt{\cosh^2 u+\sinh^2 u}}, m=<\vec{U}(u,v),\overrightarrow{x_{uv}}(u,v)>=0$$

$$n(u,v)=<\vec{U}(u,v),\overrightarrow{x_{vv}}(u,v)>=\frac{\cosh^2 u}{\sqrt{\cosh^2 u+\sinh^2 u}}$$

$$\mathcal{F}_I(u,v)=\begin{bmatrix}\sinh^2 u+\cosh^2 u & 0\\ 0 & \cosh^2 u\end{bmatrix}, \mathcal{F}_{II}(u,v)=\frac{1}{\sqrt{\cosh^2 u+\sinh^2 u}}\begin{bmatrix}-1 & 0\\ 0 & \cosh^2 u\end{bmatrix}$$

$$\mathcal{F}_I^{-1}(u,v)\mathcal{F}_{II}(u,v)=\frac{1}{\sqrt{\cosh^2 u+\sinh^2 u}}\begin{bmatrix}-\dfrac{1}{\sinh^2 u+\cosh^2 u} & 0\\ 0 & 1\end{bmatrix}$$

$$K(u,v)=\det(\mathcal{F}_I^{-1}(u,v)\mathcal{F}_{II}(u,v))=-\frac{1}{(\cosh^2 u+\sinh^2 u)^2}$$

$$H(u,v)=\frac{1}{2}tr(\mathcal{F}_I^{-1}(u,v)\mathcal{F}_{II}(u,v))=\frac{1}{2\sqrt{\cosh^2 u+\sinh^2 u}}\left(1-\frac{1}{\sinh^2 u+\cosh^2 u}\right)$$

$\because \vec{x}(0,0)=(1,0,0)$

(a) $K(0,0)=-1$

(b) $H(0,0)=0$

(c)令 $\vec{\gamma}(v)=\vec{x}(0,v)=(\cos v,\sin v,0)$

$\vec{\gamma}'(v)=(-\sin v,\cos v,0),\left\|\vec{\gamma}'(v)\right\|=1, v$ 是arclength parameter

$\vec{\gamma}''(v)=(-\cos v,-\sin v,0)$

$\kappa_n(u,v)=<\vec{\gamma}''(v),\overline{U}(u,v)>=\dfrac{\cosh u}{\sqrt{\cosh^2 u+\sinh^2 u}}$

$\kappa_n(0,0)=1$

5.Let Enneper's surface be described by the parametrized surface,

$$\vec{x}(u,v)=\left(u-\frac{u^3}{3}+uv^2, v-\frac{v^3}{3}+vu^2, u^2-v^2\right).$$

(a)What is the first fundamental form?
(b)What is the second fundamental form?
(c)What is the principal curvatures κ_1 and κ_2?
(d)Compute the Gauss map for the Enneper's surface and show that it is a one-to-one map from the Enneper's surface to the sphere.
(e)(Continued)Show that the image of the disk $\{(u,v):u^2+v^2\le 3\}$ under the Gauss map covers more than a hemisphere of the sphere.

Hint for (d) and (e): first, write the normal vector field $\overline{N}$ in polar coordinates by Letting $x=r\cos\theta$ and $y=r\sin\theta$, then focus on the third coordinate.

解：(a) $\overline{x_u}(u,v)=(1-u^2+v^2, 2vu, 2u)$

$\overline{x_v}(u,v)=(2uv, 1-v^2+u^2, -2v)$

$E(u,v)=\left\|\overline{x_u}(u,v)\right\|^2=(1+u^2+v^2)^2, F(u,v)=<\overline{x_u}(u,v),\overline{x_u}(u,v)>=0$

$G(u,v)=\left\|\overline{x_v}(u,v)\right\|^2=(1+u^2+v^2)^2$

$$\mathcal{F}_I(u,v)=\begin{bmatrix}(1+u^2+v^2)^2 & 0\\ 0 & (1+u^2+v^2)^2\end{bmatrix}$$

(b) $\overline{x_u}(u,v)\times\overline{x_v}(u,v)=(-2u(1+u^2+v^2), 2v(1+u^2+v^2), (1+u^2+v^2)(1-u^2-v^2))$

$\left\|\overline{x_u}(u,v)\times\overline{x_v}(u,v)\right\|=(1+u^2+v^2)^2$

$$\overline{U}(u,v)=\left(\frac{-2u}{1+u^2+v^2},\frac{2v}{1+u^2+v^2},\frac{1-u^2-v^2}{1+u^2+v^2}\right)$$

$\overline{x_{uu}}(u,v)=(-2u,2v,2), \overline{x_{uv}}(u,v)=(2v,2u,0), \overline{x_{vv}}(u,v)=(2u,-2v,-2)$

$l(u,v)=<\overline{U}(u,v),\overline{x_{uu}}(u,v)>=2, m=<\overline{U}(u,v),\overline{x_{uv}}(u,v)>=0$

$n(u,v)=<\overline{U}(u,v),\overline{x_{vv}}(u,v)>=-2$

$$\mathcal{F}_{II}(u,v)=\begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix}$$

(c) $\mathcal{F}_I^{-1}(u,v)\mathcal{F}_{II}(u,v)=\dfrac{2}{(1+u^2+v^2)^2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

$\therefore \kappa_1 = \dfrac{2}{(1+u^2+v^2)^2}, \kappa_2 = -\dfrac{2}{(1+u^2+v^2)^2}$

(d) $\overline{G}(u,v) = \overline{U}(u,v) = \left(\dfrac{-2u}{1+u^2+v^2}, \dfrac{2v}{1+u^2+v^2}, \dfrac{1-u^2-v^2}{1+u^2+v^2} \right)$, 令 $u = r\cos\theta, v = r\sin\theta$

$\overline{G}(u(r,\theta), v(r,\theta)) = \left(\dfrac{-2r\cos\theta}{1+r^2}, \dfrac{2r\sin\theta}{1+r^2}, \dfrac{1-r^2}{1+r^2} \right), r \geq 0, 0 \leq \theta < 2\pi$

令 $f(r) = \dfrac{1-r^2}{1+r^2}, f'(r) = \dfrac{-4r}{(1+r^2)^2} \leq 0 \Rightarrow f$ 是strictly decreasing $\Rightarrow f$ $1-1$

$\therefore \overline{G}$ 也是$1-1$

(e) $f(r) = \dfrac{1-r^2}{1+r^2}, 0 \leq r \leq \sqrt{3}$

由(d) $f(\sqrt{3}) \leq f(r) \leq f(0) \Rightarrow -\dfrac{1}{2} \leq f(r) \leq 1$

也就是$\overline{G}$ 的第3個coordinate會對應超過一半以上的sphere

6.Show that the Gaussian curvature of a rule surface

$$\vec{x}(t,v) = \overline{\alpha}(t) + v\overline{\omega}(t), t, v \in \mathbb{R},$$

is equal to zero if and only if

$$\det(\overline{\alpha}', \overline{\omega}', \overline{\omega}) \equiv 0.$$

解：$\overline{x_t}(t,v) = \overline{\alpha}'(t) + v\overline{\omega}'(t)$

$\overline{x_v}(t,v) = \overline{\omega}(t)$

$\overline{x_t}(t,v) \times \overline{x_v}(t,v) = \overline{\alpha}'(t) \times \overline{\omega}(t) + v\overline{\omega}'(t) \times \overline{\omega}(t)$

$\overline{U}(t,v) = \dfrac{\overline{\alpha}'(t) \times \overline{\omega}(t) + v\overline{\omega}'(t) \times \overline{\omega}(t)}{\left\| \overline{\alpha}'(t) \times \overline{\omega}(t) + v\overline{\omega}'(t) \times \overline{\omega}(t) \right\|}$

$\overline{x_{tt}}(t,v) = \overline{\alpha}''(t) + v\overline{\omega}''(t), \overline{x_{tv}}(t,v) = \overline{\omega}'(t), \overline{x_{vv}}(t,v) = \vec{0}$

$m(t,v) = \langle \overline{U}(t,v), \overline{x_{tv}}(t,v) \rangle = \dfrac{\langle \overline{\alpha}'(t) \times \overline{\omega}(t), \overline{\omega}'(t) \rangle}{\left\| \overline{\alpha}'(t) \times \overline{\omega}(t) + v\overline{\omega}'(t) \times \overline{\omega}(t) \right\|}, n(t,v) = 0$

$K = \det(\mathcal{F}_I^{-1}\mathcal{F}_{II}) = \dfrac{\det(\mathcal{F}_{II})}{\det(\mathcal{F}_I)} \equiv 0 \Leftrightarrow \det(\mathcal{F}_{II}) \equiv 0$

$-\dfrac{[\det(\overline{\alpha}', \overline{\omega}, \overline{\omega}')]^2}{\left\| \overline{\alpha}' \times \overline{\omega} + v\overline{\omega}' \times \overline{\omega} \right\|^2} \equiv 0 \Leftrightarrow 0 \equiv \det(\overline{\alpha}', \overline{\omega}, \overline{\omega}') = -\det(\overline{\alpha}', \overline{\omega}', \overline{\omega})$

$\therefore \det(\overline{\alpha}', \overline{\omega}', \overline{\omega}) \equiv 0$